A Tractable Approach for One-Bit Compressed Sensing on Manifolds

**Compressed Sensing**

Recover **unknown** signal \( x \in \mathbb{R}^D \) from \( m \ll D \) measurements

\[
y = Ax
\]

under following assumptions:
- \( x \) is \( s \)-sparse, i.e., at most \( s \) entries are non-zero
- measurement matrix \( A \in \mathbb{R}^{m \times D} \) is known
- \( y \in \mathbb{R}^m \) with \( y_i = (a_i, x)_j \), \( i = 1, \ldots, m \) is given.

**Recovery:** Sparcity of \( x \) allows recovery by efficient algorithms with

\[
m \geq C s \log \frac{D}{s}
\]

measurements where \( C > 0 \) is an absolute constant.

**Problem Formulation**

Recover **unknown** signal \( x \in \mathbb{R}^D \) from \( m \ll D \) one-bit measurements

\[
y = \text{sign}(Ax)
\]

under following assumptions:
- \( x \in \mathcal{M} \), where \( \mathcal{M} \) is a low-dimensional manifold of intrinsic dimension \( d \ll D \) which lies on the unit sphere \( S^{D-1} \)
- measurement matrix \( A \in \mathbb{R}^{m \times D} \) with iid Gaussian entries
- \( y \in \{-1, 1\}^m \)

**Observation:** One-bit measurements of type (2) tessellate \( S^{D-1} \) into a collection of distinguishable cells (see Figure 1).

**Definition GMRA, [1]**

Let \( J \in \mathbb{N} \) and \( k_0, k_1, \ldots, k_J \in \mathbb{N} \). For each \( j \in [J] := \{0, \ldots, J\} \) we assume to have sets \( \mathcal{E}_j \subset \mathbb{R}^D \) of centers and

\[
\mathcal{D}_j = \{ P_{j,k} : \mathbb{R}^D \to \mathbb{R}^{k_j} | k_j \in [k_j]\}
\]

of corresponding affine projectors which approximate \( \mathcal{M} \) at scale \( j \). These form a GMRA for \( \mathcal{M} \) if several assumptions (see [1]) are met.

**Reconstruction Algorithm**

**Algorithm:** One-bit Manifold Sensing (OMS)

1. **Identify** center \( c_{j,k} \) close to \( x \) via

\[
\bar{c}_{j,k} \in \arg \min_{j,k} d_{\mathcal{H}}(\text{sign}(Ac_{j,k}), y)
\]

where \( d_{\mathcal{H}} \) is the Hamming distance, i.e., \( d_{\mathcal{H}}(z, z') := |\{i : z_i \neq z'_i\}| \).

2. **If** \( d_{\mathcal{H}}(\text{sign}(Ac_{j,k}), y) = 0 \), **directly** choose \( \hat{x} = c_{j,k} \).

3. **If not,** recover the projection of \( x \) onto \( P_{j,k} \), i.e., \( P_{j,k}(x) \) by

\[
\hat{x} = \arg \min_{x \in \mathbb{R}^D} \sum_{k \in [k_j]} (\gamma_i)(a_i, x)
\]

subject to \( z \in \text{conv}(\{P_{j,k} : \mathcal{D}_j \cap \mathcal{M}, (0, 2)\}) \)

This reconstruction strategy combines

1. compressed sensing for signals on general manifolds (Iwen and Maggioni [2])
2. noisy one-bit compressed sensing (Plan and Vershynin [3]).

**Main Result**

**Notation:** Denote by \( w(A) \) the Gaussian mean width

\[
w(A) = \mathbb{E} \sup_{w \in A} |\langle g, z \rangle|, \quad g \sim N(0, I),
\]

which reflects the manifold’s complexity.

**Theorem**

There exist constants \( E, E', c > 0 \) depending on the GMRA quality such that the following holds. Let \( x \in \{0, 1/16\} \) and assume \( J \geq 10 \log(1/\sqrt{E'}) \). If

\[
m \geq E E' c^2 \max \left\{ w(A), \sqrt{d \log(1/\sqrt{E})} \right\}^2
\]

Then, with probability at least \( 1 - 12 \exp(-c^2 m) \) for all \( x \in \mathcal{M} \) the approximation \( \hat{x} \) fulfills

\[
|x - \hat{x}|^2 \leq E c.
\]

If the GMRA is created from random samples of \( \mathcal{M} \) the same result holds true with slightly changed constants and probability.

**Numerical Simulation**

Based on GMRA code provided by Mauro Maggioni, with following parameters:

- 20000 data points sampled from the 2-dimensional sphere \( \mathcal{M} \) embedded in \( \mathbb{R}^{D-1} \)
- fixed GMRA computed up to \( J = 10 \) refinement levels
- recovery of 100 randomly chosen \( x \) lying on \( \mathcal{M} \)

**References**


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