Introduction

Goal: find highly connected subgraphs in terms of edge-connectivity:
- Provide a Degeneracy-like decomposition and ensure a more efficient computation
- Find dense subgraphs and good spreaders among them

Related work:
- Degeneracy frameworks for various similar tasks already exist as the k-core and k-truss decomposition (triangle-based degeneracy for the later)
- Several functions exist in the literature that aim to solve this problem but for most formulations those functions are also hard to approximate. [H. Gabow, Journal of Computer and System Sciences '1995]
- It was shown that most efficient spreaders are located within the k-core and the k-truss of the network [Kitsak et al., Nature Physics '10]. [M. Rossi, WWW'15]

Contributions:
- An algorithm for an edge-connectivity based extension of the k-core, in order to compute it for specific k values
- Locate nodes that perform faster and wider epidemic spreading

Preliminary Concepts and Definitions

DEFINITIONS: k-core subgraph \( C_k \), Core number \( c_k \):
- \( C_k \) is k-core subgraph of \( G = (V,E) \) if it is a maximal connected subgraph in which all nodes have degree at least \( k \)
- Each node \( v \in V \) has core number \( c_v = k \), if it belongs to a k-core but not to a \((k + 1)\)-core

DEFINITION: Edge-Connectivity: A graph is \( d \)-edge connected if it has at least two vertices and for every two vertices there are \( d \) edge disjoint paths between them.

THEOREM: Menger's theorem [K. Menger, Fundamenta Mathematicae '1927']:
- Let \( x \) and \( y \) be two distinct vertices of \( G \) then the size of the minimum edge cut for \( x \) and \( y \) is equal to the number of pairwise edge-independent paths from \( x \) to \( y \).
- Extension to subgraphs: A maximal subgraph disconnected by no less than a \( k \)-edge cut is identical to a maximal subgraph with a minimum number \( k \) of edge-independent paths between any \( x, y \) pairs of nodes in the subgraph.

DEFINITIONS: k-Edge-Connectivity Core Number:
- As an application of Menger's theorem we define an edge-connectivity function in means of the minimal cut in a graph \( G \):
  \[
  \lambda(S) = \min\{|e| \in E(G) | |e| \cap S \neq \emptyset \land |e| \cap (V(G) \setminus S) \neq \emptyset\} \]
  A graph \( G \) is \( d \)-edge connected if and only if \( \lambda(G) \geq d \).
- Let \( G \) be a graph. We define the edge-connectivity degeneracy of \( G \) as a follows:
  \[
  \lambda^*(G) = \max\{\lambda(H) | H \subseteq G \}
  \]

FUNDAMENTAL RESULTS: Density and Degeneracy Inequalities: With \( v \) being \( \epsilon(G) = |E(G)| \cdot |V(G)| \), and \( \delta \) the minimal degree, \( \delta(G) = \min\{\deg(v) | v \in V(G)\} \):
- \( 2\epsilon(G) \geq \delta^2(G) \geq \lambda^*(G) \geq \epsilon(G) \)

k-Edge-Connectivity Core Algorithm Illustration (k=3)

Visualisation

Datasets

<table>
<thead>
<tr>
<th>Network name</th>
<th># Nodes</th>
<th># Edges</th>
<th>k-core</th>
<th>K-truss</th>
<th>C -</th>
<th>T</th>
<th>Epidemic threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>EMAIL-EMAIL</td>
<td>33,686</td>
<td>180,811</td>
<td>54</td>
<td>22</td>
<td>280</td>
<td>40</td>
<td>0.0084</td>
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<tr>
<td>WIKI-VOTE</td>
<td>7,066</td>
<td>100,736</td>
<td>53</td>
<td>23</td>
<td>286</td>
<td>50</td>
<td>0.0072</td>
</tr>
</tbody>
</table>

Experimental Evaluation

Method | Time Step | Final step | Max step |
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>edge core</td>
<td>21.47</td>
<td>250.93</td>
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<td>EMAIL-EMAIL</td>
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<td>204.08</td>
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<td>top degree</td>
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<td>155.48</td>
<td>2,465.60</td>
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<tr>
<td>edge core</td>
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<td>24.72</td>
<td>626.09</td>
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<td>560.66</td>
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<tr>
<td>core</td>
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<tr>
<td>top degree</td>
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<td>12.05</td>
<td>502.88</td>
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</tbody>
</table>

Table: Average number of infected nodes for some steps of the SIR model, using \( \gamma \) close to the epidemic threshold of each graph and \( \gamma = 0.8 \)

- The Edge-core method achieves higher infection rate during the first steps
- The total number of infected nodes at the end of the process is larger, while the fade out occurs earlier